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Handling Optimization under Uncertainty Problem Using Robust Counterpart Methodology

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Abstract: In this paper we discuss the robust counterpart (RC) methodology to handle the optimization under uncertainty problem as proposed by Ben-Tal and Nemirovskii. This optimization methodology incorporates the uncertain data in U a so-called uncertainty set and replaces the uncertain problem by its so-called robust counterpart. We apply the RC approach to uncertain Conic Optimization (CO) problems, with special attention to robust linear optimization (RLO) problem and include a discussion on parametric uncertainty in that case. Some new supported examples are presented to give a clear description of the used of the RC methodology theorem.

Keywords: Optimization, uncertainty, conic, robust counterpart.

Introduction

The Robust Counterpart (RC) Methodology of Ben-Tal and Nemirovskii, is one of the existing methodologies for handling uncertainty in the data of an optimization problem. Citing from Ben-Tal [9], the main challenge in this RC methodology is how and when we can reformulate the robust counterpart of uncertain problems as a computationally tractable optimization problem or at least approximate the robust counterpart by a tractable problem. Due to its definition the robust counterpart highly depends on how we choose the uncertainty set U. As a consequence we can meet this challenge only if this set is chosen in a suitable way.

A recent comprehensive survey on the works of Robust Optimization (RO) is discussed by Gabrel *et al.* [21]. The survey shows that the development concept of RO was put forward first by Mulvey *et al.* [31] who also discuss many applications. Ben-Tal and Nemirovskii [5, 6, 7, 9], and in Ben-Tal *et al.* [11] applied their RC methodology to the truss topology design (TTD) problem (Ben-Tal and Nemirovskii [4]). Later, by the same authors, many good results were obtained for robust linear optimization problems Ben-Tal and Nemirovskii [6, 7], robust quadratic and conic quadratic optimization problems Ben-Tal and Nemirovskii [10] and robust semidefinite optimization problems (see in Boyd and Vandenberghe [16]) and also together with El Ghaoui in [20].

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The RC methodology was extended to a discrete problem by Bertsimas *et al.* [12], Bertsimas and Sim [13, 14] Bertsimas and Thiele [15], Yanes and Ramirez [40], Karasan *et al.* [26], Atamturk [2] and to dynamic programming by Iyengar [24]; Nilim and El Ghaoui [33] for Markovian decision problems. RO has many applications in finance, especially in robust portfolio selection problems, Costa and Nabholz [18], Costa and Paiva [19], Goldfarb and Iyengar [22], Ito [23], robust option Lutgents and Strum [28], robust multi-stage investment (Ben-Tal *et al.* [3], Takriti and Ahmed [39]), power system capacity (Malcolm and Zenios [29]), truss topology design (Ben-Tal and Nemirovski [5]), and supply chain management (Thiele [38]).

In this paper, our aim in this paper is to overview the RC methodology in the case of uncertain conic optimization problems with a special focus onto uncertain linear optimization problems. We focus more on a discussion of a more detail proof of a crucial theorem for RC methodology of Ben-Tal Nemirovskii in case of robust linear optimization, as can be seen in Ben-Tal and Nemirovski [9]. As it is mentioned above that the main challenge of handling uncertain optimization problem is to answer the question how and when we can reformulate the robust counterpart of uncertain problems as a computationally tractable optimization problem or at least approximate the RC by a tractable problem, thus the detail proof of the crucial theorem is important to be understood since in regarding to the fast development of the theory and application of RO, a clear understanding of the RC methodology always needed. Especially for them who are new to the field of RO theory, this paper aims to give a clarification on the theorem. Some new supported examples are also presented to give a clear description of the used of the RC methodology theorem.

Methods

Optimization under Uncertainty: Topics, Difficulties and Methodologies

Optimization under uncertainty refers to the branch of optimization where the data vector ξ is uncertain. This means that the data vector ξ is not known exactly at the time when its solution has to be determined. The uncertainty may be due to measurement or modelling errors or simply to the unavailability of the required information at the time of the decision.

Consider an optimization problem of the form

$$\min_x\{f_0(x,\xi): f_i(x,\xi) \le 0, i = 1, \dots, m\} \quad (1)$$

where $f_0(x,\xi)$ denotes the objective function of the problem and the functions $f_i(x,\xi), i = 1, \dots, m$ are the constraint functions. These functions depend on the vectors x and ξ, where $x \in R^n$ is the vector of decision variables and ξ stands for the data specifying a particular problem instance.

By way of example, in the standard linear optimization problem

$$\min_x\{c^T x: Ax = b, x \ge 0\} \quad (2)$$

ξ is the triple(c, A, b) where $c \in R^n$ is the objective vector, A is the m x n constraint matrix and $b \in R^m$ is the given right-hand side vector of the constraints.

The classical approach in operations research or management science to deal with uncertainty is stochastic programming (SP) (see in Ben-Tal and Nemirovski [7]). The uncertainty in the data of the problem is then modelled by a set of random variables whose distributions are assumed to be known. A less sophisticated approach replaces each uncertain component in the data vector ξ by a representative nominal value, usually the mean value, and hence in essence ignores the uncertainty. It is also discussed in Gabrel *et al.* [21] that there are some new developments on bridging RO and stochastic programming. A discussion about RO in perspective of SP is given by Chen *et al.* in [17], the discussion focus on an introduction of an approach for constructing uncertainty sets for robust optimization using new deviation measures for random variables termed the forward and backward deviations. These deviation measures capture distributional a asymmetry and lead to better approximations of chance constraints.

We use a different approach, the aforementioned robust counterpart (RC) methodology of Ben-Tal and Nemirovskii. In this methodology, as it is mentioned above, it is assumed that the parameter data ξ belongs to an uncertainty set U.

Thus, an uncertain optimization problem can be expressed as follows:

$$\min_x\{f_0(x,\xi): f_i(x,\xi) \le 0, i = 1, \dots, m\}, \xi \in U \quad (3)$$

So it is in fact a whole family of optimization problems. One is associated with the uncertain problem (3) its so-called Robust Counterpart. In the following, we briefly discuss how this robust counterpart can be obtained. Consider the uncertain problem (3). First, we remove the uncertainty in the objective function by replacing (3) by the equivalence problem

$$\min_x\{t: f_0(x,\xi) \le t, f_i(x,\xi) \le 0, i = 1, \dots, m\}, \xi \in U \quad (4)$$

In the methodology of Ben-Tal and Nemirovskii one only considers solutions X that are feasible for this problem for all possible values $\xi \in U$ (and for some t). Thus, the set of all so-called *Robust feasible* solutions of (4) is given by

$$\{(t,x): f_0(x,\xi) \le t, f_i(x,\xi) \le 0, i = 1, \dots, m, \forall \xi \in U\} \quad (5)$$

The pair (t,x) denotes the column vector obtained by concatenating the column vectors t and x.

Now the robust counterpart of the uncertain problem (4) consists of minimizing$x$ over this set:

$$\min_x\{t: f_0(x,\xi) \le t, f_i(x,\xi) \le 0, i = 1, \dots, m, \forall \xi \in U\} \quad (6)$$

Obviously, the robust counterpart of (4) represents a worst-case oriented approach: a pair of solutions (t,x) is robust feasible only if x satisfies the constraints for all possible values of ξ (and some t). The optimal solutions of (6) are called robust optimal solutions. Note that the robust counterpart (6) is an optimization problem with usually infinitely many constraints, depending on the uncertainty set U. This implies that this problem may be very hard to solve. This means that only if U is chosen suitably, the problem (6) can be solved efficiently.

Uncertain Conic Problem

In this section we discuss one of the important optimization class problems, i.e, Conic optimization (CO). This class of problem is a very useful optimization technique that concerns the problem of minimizing a linear objective function over the intersection of an affine set and a convex cone. The importance of this class of problems is due to two facts, i.e., many practical nonlinear problems can be modeled as a CO problem, and a wide class of CO problems can be solved efficiently by so-called interior-point methods.

The interest in CO was highly stimulated when it became clear that the interior-point methods that were developed in the two last decades for Linear Optimization (LO) see, e.g., Jarre [25], Mehrotra [30], Nesterov and Nemirovskii [32], Peng *et al.* [34], Renegar [35], Roos *et al.* [36], Sturm and Zhang [37]) and which revolutionized the field of LO, could be naturally extended to obtain polynomial-time methods for CO. The most elegant theory developed by Nesterov and Nemirovskii [32] provides an interior-point method with polynomial complexity if the underlying cone has a so-called *self-concordant barrier* that is computationally tractable.

This opened the way to a wide spectrum of new applications which cannot be captured by LO, e.g., in image processing, finance, economics, control theory, combinatorial optimization, etc. For a nice survey both of the theory of CO and many new applications, we refer to the book of Ben-Tal and Nemirovskii [8]; Ben-Tal *et al.* [11].

In this paper we do not touch the algorithmic aspects of interior-point methods for CO. We refer the interested reader to the existing literature, where one can find a wide variety of such methods. See, e.g., the above references and some numerical evidences for the efficiency of these methods has been provided by many authors (see in Andersen *et al.* [1], Jarre [25], Karasan *et al.* [27], Mehrotra [30], Peng *et al.* [34], Sturm and Zhang [37]).

The general form of a conic optimization problem is as follows:

$$\min_{x\in R^n}\{c^T x : A_i x - b_i \in \kappa^i, i = 1, \dots, m\} \quad (7)$$

where the objective function is $c^T x$, with $c \in R^n$. Furthermore $A_i x - b_i$ represents an affine function from R^n to R^m. Each κ^i denotes convex cones in R^m, it is either a non-negative orthant (linear constraints) or a Lorentz cone (conic quadratic constraints), or a semidefinite cone (linear matrix inequalities).

The easiest and most well-known case occurs whenthe cone κ^i is the nonnegative orthantof R^m, i.e., when $\kappa^i = R^m_+$. Then the above problem gets the form

$$\min_{x\in R^n}\{c^T x : A_i x - b_i \in R^m_+\} \quad (8)$$

This is nothing but one of the standard forms of the well-known LO problem. Thus it becomes clear that

LO is a special case of CO. When the data associated with (7), i.e., the triple $(c, \{A_i, b_i\}_{i=1}^m)$, is uncertain and is only known to belong to some uncertainty set U, we speak about the uncertain conic problem which has the following form:

$$\min\{c^T x : A_i x - b_i \in \kappa^i, i = 1, \dots, m\},$$
$$(c, \{A_i, b_i\}_{i=1}^m) \in U \quad (9)$$

The robust counterpart to (8) is the following convex problem

$$\min_{t\in R, x\in R^n}\{t : c^T x \le t, A_i x - b_i \in \kappa^i, i = 1, \dots, m : \forall (c, \{A_i, b_i\}_{i=1}^m) \in U\} \quad (10)$$

This is a CO problem with usually infinitely many constraints, depending on the uncertainty set U. Hence, in general, this problem may be very hard to solve. In the next section, we discuss the robust linear optimization problem and we show that for special choices of the uncertainty set U the problem (10) is computationally tractable.

Results and Discussion

Robust Linear Optimization and Its Examples

An uncertain linear optimization problem has the following form:

$$\min_x\{c^T x : Ax \ge b\}, (c, A, b) \in U \quad (11)$$

Where U is the set of all possible realizations of (c, A, b), with (each) matrix A having size mxn. It is important to be mentioned here that the set of parameter $(c, A, b)$is not a random variables. The set U will be the representation of the uncertain (c, A, b). To this end, the following discussion will give some explanation on it. As mentioned in the previous section, the first step to do the RC methodology is removing the uncertain from the objective function. This implies that the robust counterpart of (11) is the following semi-infinite optimization problem:

$$\min_{t,x}\{t : t \ge c^T x, Ax - b \ge 0, \forall (c, A, b) \in U\} \quad (12)$$

The tractability (12) depends on the uncertainty set U. The following theorem makes clear that if the set U can be described either by linear constraints, or conic quadratic constraints or by a semidefinite constraint, then (12) becomes computationally tractable. Because the following theorem is crucial for the paper, and since the proof in Ben-Tal and Nemirovski [9] is written with less detail, we include a more detailed proof below.

Theorem 1 (Ben-Tal and Nemirovski [9])

Assume that the uncertainty set U in (11) is given as the affine image of a bounded set $Z = \{\xi\} \subset R^N$, and Z is given either by a system of linear inequalities $P\xi \le p$, or

1. by a system of conic quadratic inequalities $\|P_i\xi - p_i\|_2 \le q_i^T\xi - r_i, i = 1, \dots, M$ or

2. by a linear matrix inequality $P_0 + \sum_{i=1}^{N} \xi_i P_i \succcurlyeq 0$ where P, P_i, P_0 are matrices and p, p_i, q_i are vectors.

In the cases 2 and 3 we assume that the system of constraints defining U is strictly feasible. Then the robust counterpart (12) of (11) is equivalent to

1. a linear optimization problem in case 1,
2. a conic quadratic problem in case 2,
3. a semidefinite problem in case 3.

In all cases, the data of the resulting robust counterpart problem are readily given by m, n equation reference goes here and the data specifying the uncertainty set. Moreover, the size of the resulting problem is polynomial in the size of the data specifying the uncertainty set.

Proof:

By assumption, the uncertainty set U has the following form

$$U = \left\{(c, A, b): (c, A, b) = (c^0, A^0, b^0) + \sum_{j=1}^{N} \xi_j \, (c^j, A^j, b^j), \xi \in Z\right\} \quad (13)$$

where (c^0, A^0, b^0) is a nominal data vector, $c^j \in R^n, A^j \in R^{m\text{x}n}$ and $b^j \in R^m$. The feasible set of (12) is

$$F = \{(t, x): c^T x \leq t, Ax - b \geq 0, \forall (c, A, b) \in U\} \quad (14)$$

where with U as given by (13). This implies that the pair (t, x) is robust feasible if and only if

$$t \geq \left(c^0 + \sum_{j=1}^{N} \xi_j c^j\right)^T x, \left(A^0 + \sum_{j=1}^{N} \xi_j A^j\right) - \left(b^0 + \sum_{j=1}^{N} \xi_j b^j\right) \geq 0, \forall \xi \in Z \quad (15)$$

Now, let C be the $n \, \text{x} \, N$ matrix with columns $c^1, \dots, c^N$. Thus, the first constraint in (15) can be rewritten as follows.

$$\begin{aligned} 0 &\leq t - \left(c^0 + \sum_{j=1}^{N} \xi_j c^j\right)^T x \\ &= t - (c^0 + \xi_1 c^1 + \cdots + \xi_N c^N)^T x \\ &= t - (c^0 + C\xi)^T x' \end{aligned} \quad (16)$$

This equivalent to

$$0 \leq \left(\begin{bmatrix} 1 \\ -c^0 \end{bmatrix} + \begin{bmatrix} 0 \\ -C \end{bmatrix} \xi\right)^T \begin{bmatrix} t \\ x \end{bmatrix} \quad (17)$$

Letting A_i^j denote the i-th row of the matrix A^j, for $j = 0, \dots, N$ and $i = 1, \dots, m$, the second constraint in (14) can be written as

$$\begin{aligned} 0 &\leq \left(A_i^0 + \sum_{j=1}^{N} \zeta_j A_i^j\right) x - \left(b_i^0 + \sum_{j=1}^{N} \zeta_j b_i^j\right) \\ &= A_i^0 x + (\zeta_1 A_i^1 + \cdots + \zeta_N A_i^N) x \\ &\quad - (b_i^0 + \zeta_1 b_i^1 + \cdots + \zeta_N b_i^N) \end{aligned} \quad (18)$$

Let P_i denote the $n \, \text{x} \, N$ matrix with columns $A_i^1, \dots, A_i^N$ and d_i the column vector with entries $b_i^1, \dots, b_i^N$. Then the i-th inequality in (18) is equivalent to

$$\begin{aligned} 0 &\leq A_i^0 x + (P_i \xi)^T x - (b_i^0 + d_i^T \xi) \\ &= \left(\begin{bmatrix} 0 \\ (A_i^0)^T \end{bmatrix} + \begin{bmatrix} 0 \\ P_i \end{bmatrix} \xi\right)^T \begin{bmatrix} t \\ x \end{bmatrix} - (b_i^0 + d_i^T \xi) \end{aligned} \quad (19)$$

From (17) and (19), the pair $y = (t, x) \in R^{n+1}$ is a feasible solution of the RC (12) if and only if y satisfies

$$[D_i \zeta + \beta_i]^T y - [f_i^T \xi + \delta_i] \geq 0, \forall \zeta \in Z, i = 0, \dots, m \quad (20)$$

where

$$D_0 = \begin{bmatrix} 0 \\ -C \end{bmatrix}, \beta_0 = \begin{bmatrix} 1 \\ -c^0 \end{bmatrix}, d_0^T = [0, \dots, 0], \delta_0 = 0 \quad (21)$$

and, for $i = 1, \dots, m$ the following holds.

$$D_i = \begin{bmatrix} 0 \\ P_i \end{bmatrix}, \beta_i = \begin{bmatrix} 0 \\ (A_i^0)^T \end{bmatrix}, d_i^T = [b_i^1, \dots, b_i^N], \delta_i = b_i^0 \quad (22)$$

This holds if and only if y is such that the optimization problem

$$\inf_\xi \{(\beta_i + D_i \xi)^T y - (\delta_i + d_i^T \xi): \xi \in Z\} \quad (23)$$

has a nonnegative optimal value, for each $i = 1, \dots, m$. Now, by assumption, $Z = \{\xi\} \subset R^N$ is representable as

$$Z = \{\xi: R\xi - \rho \in \kappa\} \quad (24)$$

for suitable R and ρ, where κ is either a nonnegative orthant, or a direct product of the second order cones, or the semidefinite cone. Consequently, $(G_i[y])$ can be written as follows:

$$\inf_\xi \{(D_i^T y - d_i)^T \xi + (\beta_i^T y - \delta_i): R\xi - \rho \in \kappa\} \quad (25)$$

Note that this conic problem is bounded (since Z is bounded) and κ is a self-dual cone. Moreover, if the cone κ is nonlinear, then the problem is strictly feasible, by the assumption for the cases 2 and 3 in the theorem. Therefore, the optimal value of the problem is equal to the optimal value of the dual problem, by the Strong Conic Duality in Ben-Tal and Nemirovskii [8], and the dual problem is solvable. Note that the term $\beta_i^T y - \delta_i$ in the objective function does not depend on ξ and hence it can be considered as a constant. Introducing a vector of dual variables ξ^i, the dual problem of (25) is the following conic problem:

$$\max_{\xi^i} \{\rho^T \xi^i + \beta_i^T y - \delta_i : R^T \xi^i = D_i^T y - d_i, \xi^i \in \kappa\} \quad (26)$$

Since $(H_i[y])$ and $(G_i[y])$ have the same optimal value, we may conclude that y is robust feasible if and only if the optimal value of $(H_i[y])$ is nonnegative for each i. At this stage we use that each $(H_i[y])$ is solvable, i.e. has an optimal solution. We may therefore conclude that y is robust feasible for each i there exists a vector ξ^i such that

$$\rho^T\xi^i + \beta_i^T y - \delta_i \geq 0, R^T\xi^i = D_i^T y - d_i, \xi^i \in \kappa \qquad (27)$$

We conclude from this that the robust counterpart (2) of (1) is equivalent to the problem

$$\min_{y=(t;x),(\xi^i)_{i=1}^m} \{t: \rho^T\xi^i + \beta_i^T y - \delta_i \geq 0,$$
$$R^T\xi^i = D_i^T y - d_i, \xi^i \in \kappa, i = 0, \dots, m\} \qquad (28)$$

This is a linear optimization problem if the cone κ is linear, a second-order cone problem if κ is a direct product of second-order cone(s), and a semidefinite problem if the cone κ is a semidefinite cone. Hence the proof is complete.■

In the next subsection, we present some new simple examples to illustrate the use of Theorem 1.

Some Examples to Illustrate the Use of Theorem 1

We start by considering the following uncertain problem:

$$\min_x\{-x: (\xi - 1)x \leq 1, 0 \leq x \leq 1\}, |\xi| \leq r \qquad (29)$$

Where r is a given nonnegative number. Obviously the uncertainty is only in the constraint matrix, due to the uncertain parameter ξ. We derive the RC of this problem just by applying the method outlined in the proof of Theorem 1.

In the current case we have

$$c = [-1], A = \begin{bmatrix} 1-\zeta \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \zeta \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \qquad (30)$$

So we have $m = 3, n = 1$ and $N = 1$. Using the notations of (13), the set U is defined by

$$c^0[-1], c^1 = [0], A^0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, A^1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$
$$b^0 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (31)$$

In the present case the n x N matrix C is given by= c^1. Hence we have

$$D_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \beta_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, d_0 = [0], \delta_0 = 0 \qquad (32)$$

The n x N matrices P_i for $i = 1, \dots, m$ are given by

$$P_1 = [-1], P_2 = [0], P_3 = [0] \qquad (33)$$

whence it follows that

$$D_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, D_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (34)$$

Furthermore,

$$\beta_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \beta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \beta_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad (35)$$

and since b^1 is the zero vector,

$$d_1 = d_2 = d_3 = [0] \qquad (36)$$

Using the entries in b^0,

$$\delta_1 = \delta_2 = \delta_3 = -1 \qquad (37)$$

To proceed we need to find a conic representation of the set

$$Z = \{\xi: |\xi| \leq r\} \qquad (38)$$

To keep things simple we observe that this set allows a linear description as follows:

$$Z = \{\xi: -r \leq \xi \leq r\} = \{\xi: R\xi - \rho \in R_+^2\} \qquad (39)$$

where

$$R = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \rho = \begin{bmatrix} -r \\ -r \end{bmatrix} \qquad (40)$$

We only have to substitute all the computed entities in (28) to obtain the robust counterpart of the given problem. We do this in steps. Note that since $m = 3$, we have constraints for $i = 0, \dots, 3$ and these are given by

$$\rho^T\xi^i + \beta_i^T y - \delta_1 \geq 0, R^T\xi^i = D_i^T y - d_i, \xi^i \in R_+^2 \qquad (41)$$

where $y = (t; x)$. The constraints for the respective values of$i$ are:

$$i = 0: \begin{bmatrix} -r \\ -r \end{bmatrix}^T \xi^0 + \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} t \\ x \end{bmatrix} \geq 0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T \xi^0 = 0, \xi^0 \geq 0 \qquad (42)$$

$$i = 1: \begin{bmatrix} -r \\ -r \end{bmatrix}^T \xi^1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} t \\ x \end{bmatrix} + 1 \geq 0,$$
$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}^T \xi^1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}^T \begin{bmatrix} t \\ x \end{bmatrix}, \xi^1 \geq 0 \qquad (43)$$

$$i = 2: \begin{bmatrix} -r \\ -r \end{bmatrix}^T \xi^2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} t \\ x \end{bmatrix} \geq 0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T \xi^2 = 0, \xi^2 \geq 0 \qquad (44)$$

$$i = 3: \begin{bmatrix} -r \\ -r \end{bmatrix}^T \xi^3 + \begin{bmatrix} 0 \\ -1 \end{bmatrix}^T \begin{bmatrix} t \\ x \end{bmatrix} + 1 \geq 0,$$
$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}^T \xi^3 = 0, \xi^3 \geq 0 \qquad (45)$$

In other words,

$$i = 0: -r(\xi_1^0 + \xi_2^0) + t + x \geq 0, -\xi_1^0 + \xi_2^0 = 0, \xi^0 \geq 0 \qquad (46)$$

$$i = 1: -r(\xi_1^1 + \xi_2^1) + x + 1 \geq 0,$$
$$-\xi_1^1 + \xi_2^1 = -x, \xi^1 \geq 0 \qquad (47)$$

$$i = 2: -r(\xi_1^2 + \xi_2^2) + x \geq 0,$$
$$-\xi_2^2 + \xi_2^2 = 0, \xi^2 \geq 0 \qquad (48)$$

$$i = 3: -r(\xi_1^3 + \xi_2^3) - x + 1 \geq 0,$$
$$-\xi_1^3 + \xi_2^3 = 0, \xi^3 \geq 0 \qquad (49)$$

Yet we observe that only the variable t appears in the objective function of (27). Since $r \geq 0$ the constraints for $i = 0$ can be satisfied if and only if $t + x \geq 0$.

Similarly, the constraints for $i = 2$ can be satisfied if and only if $x \geq 0$ and those for $i = 3$ if and only if $1 - x \geq 0$. Replacing ξ_1^1 by α and ξ_2^1 by β, we get the following equivalent system of constraints:

$i = 0: t \geq -x$ (50)
$i = 1: -r(\alpha + \beta) + x + 1 \geq 0, -\alpha + \beta = x,$
$\alpha \geq 0, \beta \geq 0$ (51)
$i = 2: x \geq 0$ (52)
$i = 3: 1 - x \geq 0$ (53)

We can also simplify the constraints for $i = 1$. Note that$\alpha = x + \beta$ is automatically nonnegative, since x and β are nonnegative. So the constraint $\alpha \geq 0$ is redundant. Hence, by eliminating α we get the equivalent system

$-r(x + 2\beta) + x + 1 \geq 0, \beta \geq 0$ (54)
or, equivalently,
$(1 - r)x \geq -1 + 2r\beta, \beta \geq 0$ (55)

Using again that $r$is nonnegative we see that if the first inequality is satisfied for some nonnegative β then it is also also satisfied for$\beta = 0$. Hence the constraints for $i = 1$ may be replaced by the single constraint$(1 - r)x \geq -1$. Thus we obtain the following RC of the given problem:

$\min_{t,x}\{t: t \geq -x, 0 \leq x \leq 1, (1 - r)x \leq 1\}$ (56)

By eliminating the variable t we get the following problem, which has the same optimal value, denoted as $Opt(r)$.
$Opt(r) = \min_{t,x}\{-x, 0 \leq x \leq 1, (r - 1)x \leq 1\}$ (57)

Note that if$x = 1$ is feasible, then it is the optimal solution. This happens for $r \leq 2$. If $r > 2$ then the largest possible value for x is $1/(r - 1)$. Hence we have

$$Opt(r) = \begin{cases} -1, \text{ for } 0 \leq r \leq 2 \\ -\frac{1}{r-1} \text{for } r > 2 \end{cases} \quad (58)$$

The graph of $Opt(r)$ is as shown in Figure 1.

Note that $Opt(r)$ depends continuously on the parameter r. As we will see in the next example this is not always the case. The reader may have noticed that the RC in the above simple example could have been obtained much easier by noting that the `worst' value of ξ in (28) occurs when $\xi = r$. This immediately yields the RC of (29).

In the next example we derive the RC in this more direct way. The aim of the above example, however, was to demonstrate how the RC of a problem that satisfies the hypothesis of Theorem 1 can be obtained in a straightforward way by using the scheme presented in the proof of Theorem 1.

Example 2
For $r \geq 0$, consider the uncertain problem:
$\min_x \left\{-x: (\xi - 1)x \leq 1, \frac{1}{2} \leq x \leq 1\right\}, |\xi| \leq r$ (59)

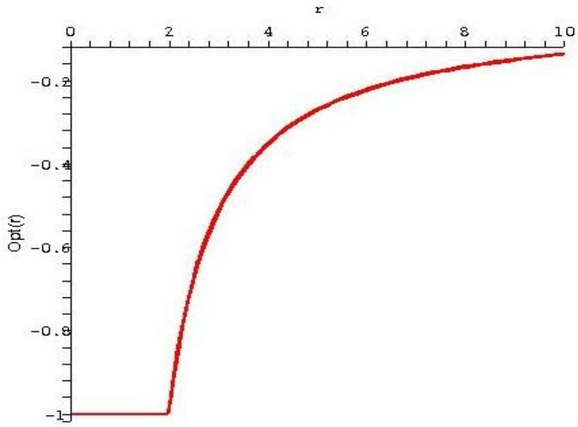

Figure 1. The robust optimal value function of $Opt(r)$of Example 1.

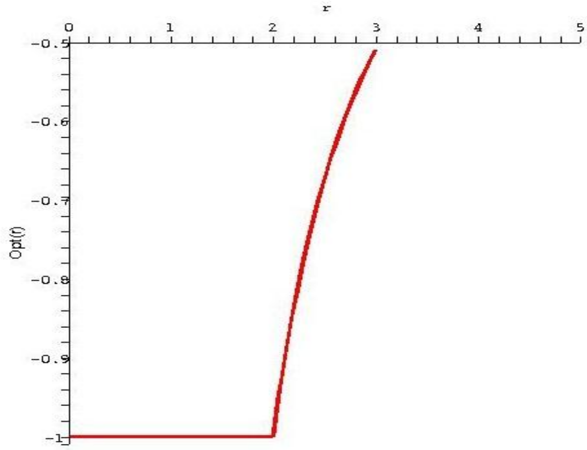

Figure 2. The robust optimal value function of $Opt(r)$of Example 2.

As in the previous example the `worst' value of ξ occurs when $\xi = r$. Hence, the RC is given by

$\min_x \left\{-x: (r - 1)x \leq 1, \frac{1}{2} \leq x \leq 1\right\}$ (60)

As in the first example, $x = 1$ is feasible if $r \leq 2$ and then this is the robust optimal solution. If $r > 2$ then the first constraint becomes

$x \leq \frac{1}{r-1}$ (61)

which makes clear that the problem is infeasible if $r > 3$. Hence we have

$$Opt(r) = \begin{cases} -1 & \text{for } 0 \leq r \leq 2, \\ -\frac{1}{r-1} & \text{for } 2 \leq r \leq 3, \\ \infty & \text{for } r > 3 \end{cases} \quad (62)$$

See also Figure 2.

Conclusion

From the discussion above, we may conclude the paper by claiming that RC methodology can be employ to obtain the robust optimal solution of uncertain CO as long as the RC formulation can be represented in CO formulation wheather it is a linear, conic quadratic or semidefininte optimization. From the discussed examples, it can be conclude that the optimal value of RC which is denotes as $Opt(r)$ is not always depends continuously on the parameter r.

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